THE
KING'S
SCHOOL

2021

Year 12
Alternative Higher School Certificate Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A Reference Sheet is provided
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I 10 marks

- Attempt Questions 1-10
- Use the Multiple Choice answer sheet provided
- Allow about 15 minutes for this section

Section II 60 marks

- Attempt Questions 11-14
- Allow about 1 hour and 45 minutes for this section

Examiners' Use Only **Setter/Checker: External/GJD**

Question	Calculus	Functions	Combinatorics	Statistical Analysis	Trigonometric Functions	Vectors	Total
Section I							
1-10	4, 6, 9,10 / 4	2,3,7 / 3	8 /1		1 / 1	5 / 1	**/10**
Section II							
11		a,b /5	d / 4		c / 2	e /4	**/15**
12	d / 3	a,e / 5		c /3		b / 4	**/15**
13	a,b,d / 9				c / 6		**/15**
14		c 6	a /4			b /5	**/15**
Total	**/16**	**/19**	**/ 9**	**/3**	**/ 9**	**/ 14**	**/70**

Section 1

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

---

1 Which of the following expressions is equivalent to $3\cos x - \sqrt{3}\sin x$?

(A) $2\sqrt{3}\cos\left(x+\frac{\pi}{3}\right)$

(B) $2\sqrt{3}\cos\left(x-\frac{\pi}{3}\right)$

(C) $2\sqrt{3}\cos\left(x+\frac{\pi}{6}\right)$

(D) $2\sqrt{3}\cos\left(x-\frac{\pi}{6}\right)$

2 Which of the following is the correct expansion of $(2x-1)^3$?

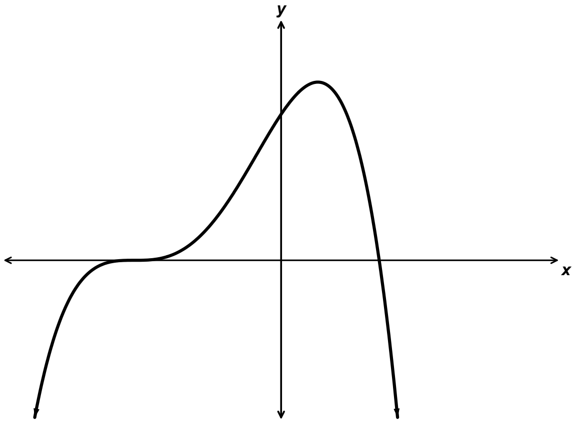
(A) $8x^3+12x^2+6x-1$

(B) $8x^3-12x^2+6x-1$

(C) $8x^3-4x^2+2x-1$

(D) $2x^3-12x^2+6x-1$

3 The graph of a polynomial $y = P(x)$ is shown below.



What is the minimum possible degree of the polynomial $[P(x)]^2$?

(A) 2

(B) 4

(C) 8

(D) 16

4 Which of the following is the derivative of $y = x^2 \sin^{-1}(2x)$?

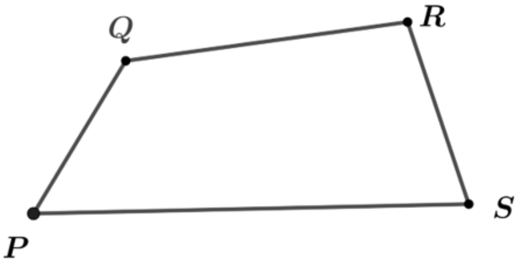
(A) $2x\sin^{-1}(2x) + \dfrac{2x^2}{\sqrt{1-4x^2}}$

(B) $2x\sin^{-1}(2x) + \dfrac{x^2}{\sqrt{1-4x^2}}$

(C) $2x\sin^{-1}(2x) + \dfrac{2x^2}{\sqrt{1-2x^2}}$

(D) $2x\sin^{-1}(2x) + \dfrac{x^2}{\sqrt{1-2x^2}}$

5 $PQRS$ is a quadrilateral where $\overrightarrow{PQ}=\overrightarrow{p}$, $\overrightarrow{QR}=\overrightarrow{q}$, $\overrightarrow{RS}=\overrightarrow{r}$ and $\overrightarrow{SP}=\overrightarrow{s}$



Which of the following is true?

(A) $\overrightarrow{p}+\overrightarrow{q}=\overrightarrow{s}+\overrightarrow{r}$

(B) $\overrightarrow{p}+\overrightarrow{q}=-\left(\overrightarrow{s}+\overrightarrow{r}\right)$

(C) $\overrightarrow{p}+\overrightarrow{s}=\overrightarrow{q}+\overrightarrow{r}$

(D) $\overrightarrow{p}+\overrightarrow{s}=-\left(\overrightarrow{q}-\overrightarrow{r}\right)$

6 The slope field for a first order differential equation is shown below.

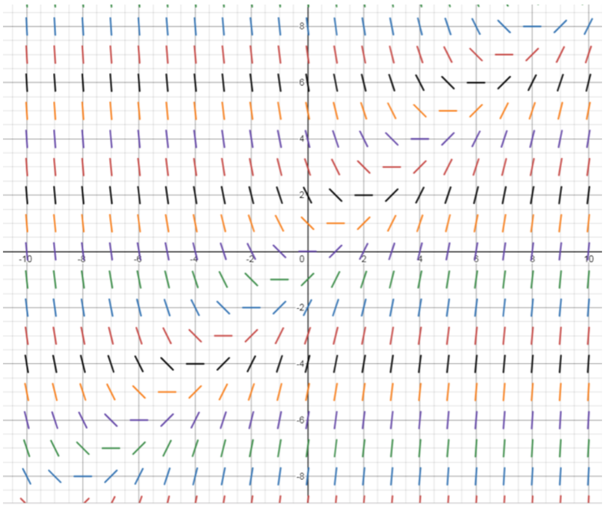
Which of the following could be the differential equation represented?

(A) $\dfrac{dy}{dx}=x+y$

(B) $\dfrac{dy}{dx}=x-y$

(C) $\dfrac{dy}{dx}=y-x$

(D) $\dfrac{dy}{dx}=-xy$



7 Which of the following is the solution set of $\dfrac{2}{1-3^x} > -1$?

(A) $x < 0$

(B) $x < 1$

(C) $0 < x < 1$

(D) $x < 0$ or $x > 1$

8 A bag contains 5 identical blue marbles, 6 identical black marbles and 3 identical red marbles. Three marbles are drawn at random.

Which expression below gives the correct probability that exactly two blue marbles are drawn?

(A) $\dfrac{{}^{5}C_2}{{}^{14}C_3}$

(B) $\dfrac{{}^{5}C_2 \times {}^{9}C_1}{{}^{14}C_3}$

(C) $\dfrac{2}{5} \times \dfrac{1}{9}$

(D) $\dfrac{2}{14} \times \dfrac{1}{9}$

9 Which of the following is the value of $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{4-4x^2}}\,dx$?

(A) $\frac{1}{4}$

(B) $\frac{\pi}{12}$

(C) $\frac{\pi}{6}$

(D) 1

10 The population, *P*, of animals, in an environment in which there are scarce resources, is increasing such that $\frac{dP}{dt} = P(100-P)$, where *t* is time.

The initial population is 20 animals.

Which of the following is true?

(A) $P = 100 - 80e^{100t}$

(B) The population is increasing most rapidly when $P = 50$.

(C) The population is increasing most rapidly when $t = 50$.

(D) The maximum population is $P = 50$.

Section II

60 marks

Attempt Questions 11 – 14

Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

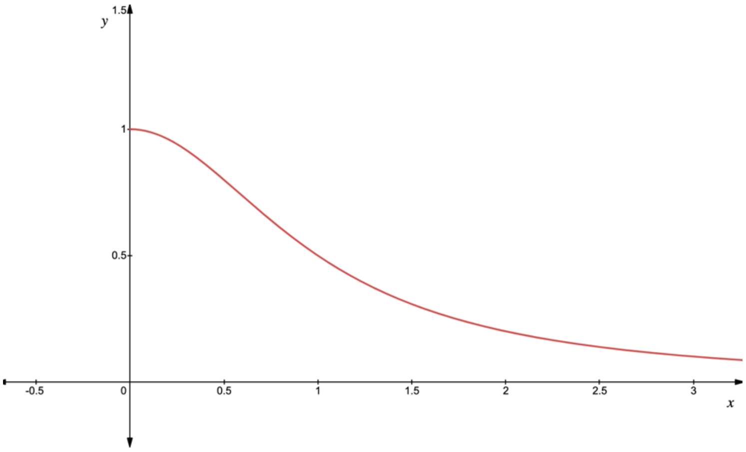
Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Given $P(x)=x^3-5x^2+8x-4$,

(i) Find $P(1)$. **1**

(ii) Factorise $P(x)$ **2**

(b) **2**



The diagram above shows the function $f(x)=\dfrac{1}{x^2+1}$ for $x\geq 0$.

State the domain and range of the inverse function $f^{-1}(x)$.

(c) Given $\cos x=-\dfrac{1}{3}$ and $\sin x>0$ find the exact value of $\sin 2x$. **2**

Question 11 continues on page 8

Question 11 (continued)

(d) Ten members of a sporting team have the numbers 1 to 10 on the back of their jumpers.

- (i) Explain why, if six players are randomly chosen, there must be at least one pair of players whose numbers add to 11. **2**
- (ii) What is the minimum number of players that need to be chosen to ensure that at least one pair of players have numbers that add to 12? **2**

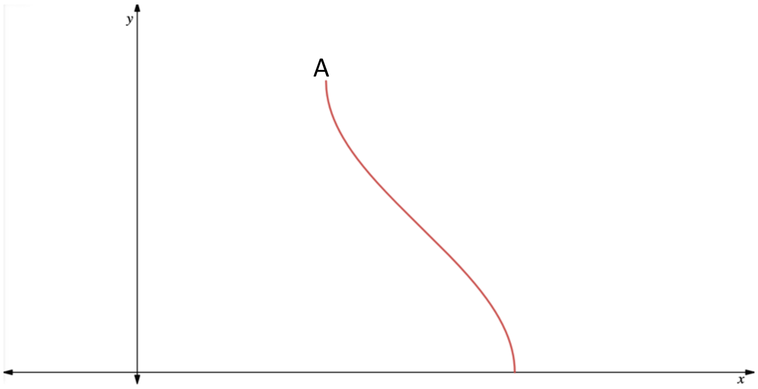
(e) A projectile is fired from horizontal ground such that horizontal and vertical displacements (in metres) from the point of projection, at time t seconds, are given by $x = \left(30\sqrt{3}\right)t$ and $y = 30t - 5t^2$.

- (i) Find the angle at which the projectile was fired. **3**
- (ii) Find the speed at which the projectile was fired. **1**

Question 12 on page 9

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) **2**



The graph above shows $y = 2\cos^{-1}(2x-3)$.

Find the coordinates of point A.

(b) (i) The vectors $\vec{p} = \begin{pmatrix} a \\ 3 \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ are parallel. **1**

Find a.

(ii) The vector $\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ is perpendicular to vector $\vec{q}$ and **3**

$\left|\vec{q} - \vec{r}\right| = \sqrt{65}$. Find the possible values of x and y.

(c) The probability that a certain drug cures a disease is 0.8.The drug is administered to 10 000 patients.

Let X be the binomial random variable representing the number of patients cured.

(i) Find $E(X)$. **1**

(ii) Show that X has a standard deviation of 40. **1**

(iii) Use a normal approximation to find the probability that X is less than 8040. **1**

Question 12 continues on page 10

Question 12 (continued)

(d) A curve has gradient given by $\frac{dy}{dx} = 2xy$. Find an expression for **3**

y in terms of x given that the curve passes through the point $(0,4)$

(e) Use mathematical induction to prove **3**

$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + ... + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$ for integers $n \geq 1$

Question 13 on page 11

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int_0^{\frac{\pi}{2}} \sin^2 x \cos x \, dx$ **2**

(b) Use the substitution $x = 2\sin\theta$ to find $\int_0^1 \sqrt{4-x^2}\, dx$. **3**

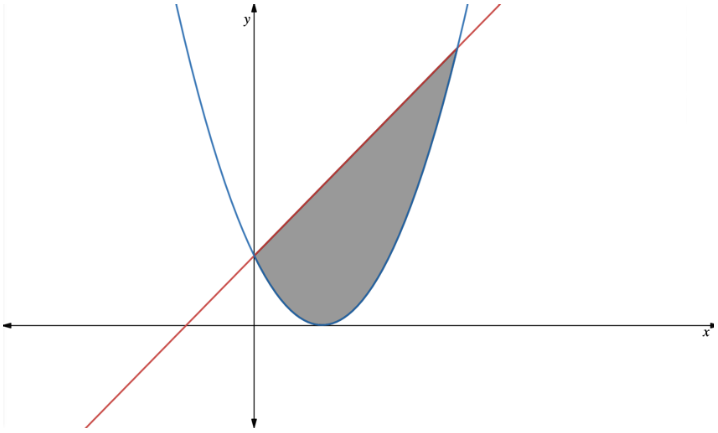
(c) (i) Show $\sin(x-y)\sin(x+y) = \sin^2 x - \sin^2 y$. **3**

(ii) Solve $\sin^2 3x - \sin^2 x = \sin 4x$ for $0 \le x \le \pi$ **3**

(d) The shaded area in the diagram below is bounded by $y = (x-1)^2$ **4**

and $y = x+1$. The area is rotated around the x-axis to form a solid.

Find the volume of the solid generated.



Question 14 on page12

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Use the identity

$$(1+x)^{n+m} = (1+x)^n\,(1+x)^m$$

to show that, if m > n, **2**

$$\binom{n}{0}\binom{m}{n}+\binom{n}{1}\binom{m}{n-1}+\binom{n}{2}\binom{m}{n-2}+\cdots\binom{n}{n}\binom{m}{0}=\binom{m+n}{n}$$

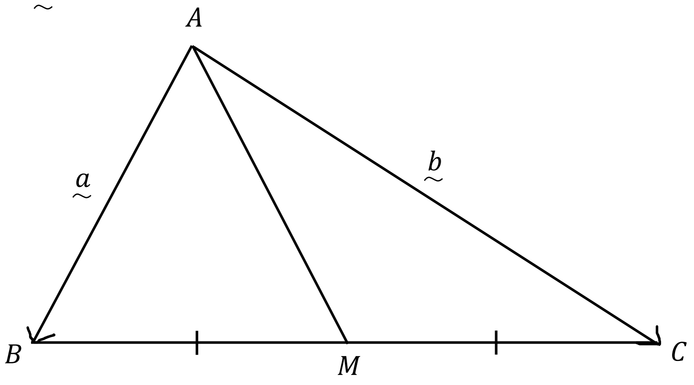
(ii) Simon has n white canaries and m yellow canaries, where $m > n$. **2**

He wants to move *n* canaries, with at least one of each colour, to a new cage.

What is the number of ways this could be done?

(b) In the diagram, M is the midpoint of the side BC of triangle ABC.

Let $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{AC} = \underset{\sim}{b}$.



Copy the diagram.

(i) Find the vectors $\overrightarrow{AM}$ and $\overrightarrow{CM}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **2**

(ii) Show that $|\overrightarrow{AM}|^2+|\overrightarrow{CM}|^2=\frac{1}{2}\left(|\overrightarrow{AB}|^2+|\overrightarrow{AC}|^2\right)$ **3**

(c) (i) Show that $8\cos^4\theta-8\cos^2\theta+1-\cos 4\theta=0$. **2**

(ii) By letting $x=\cos\theta$ in the equation $16\,x^4-16\,x^2+2-\sqrt{2}=0$, **2**

show that $\cos 4\theta=\frac{\sqrt{2}}{2}$.

(iii) Prove that $cos^2\frac{\pi}{16}\,cos^2\frac{7\pi}{16}=\frac{2-\sqrt{2}}{16}$. **2**

End of Examination

SUGGESTED SOLUTIONS 2021 Mathematics Extension 1 Trial HSC Examination

Section 1

10 marks

Questions 1 – 10 (1 mark each)

Question 1 (1 mark)

Outcomes assessed: ME11-3

Targeted Performance Bands: E2

Solution	Answer	Mark
$2\sqrt{3}\cos\left(x+\frac{\pi}{6}\right)=2\sqrt{3}\left(\cos x\cos\frac{\pi}{6}-\sin x\sin\frac{\pi}{6}\right)$ $=2\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos x-\frac{1}{2}\sin x\right)$ $=3\cos x-\sqrt{3}\sin x$	**C**	**1**

Question 2 (1 mark)

Outcomes assessed: ME11-2

Targeted Performance Bands: E2-E3

Solution	Answer	Mark
$(2x-1)^3=(2x+(-1))^3$ $=(2x)^3+3(2x)^2(-1)+3(2x)(-1)^2+(-1)^3$ $=8x^3-12x^2+6x-1$	**B**	**1**

Question 3 (1 mark)

Outcomes Assessed: ME11-2

Targeted Performance Bands: E2-E3

Solution	Answer	Mark
Minimum degree of $P(x)$ is 4 $\therefore$ Minimum degree of $\left[P(x)\right]^2$ is $2\times4=8$	**C**	**1**

Question 4 (1 mark)

Outcomes assessed: ME12-1

Targeted Performance Bands: E2

Solution	**Answer**	**Mark**
$y = x^2 \sin^{-1}(2x)$ $\frac{dy}{dx} = 2x\left(\sin^{-1}(2x)\right) + x^2\left(\frac{1}{\sqrt{1-(2x)^2}} \times 2\right)$ $= 2x\sin^{-1}(2x) + \frac{2x^2}{\sqrt{1-4x^2}}$	**A**	**1**

Question 5 (1 mark)

Outcomes assessed: ME12-2

Targeted Performance Bands: E3

Solution	**Answer**	**Mark**
$\overrightarrow{p} + \overrightarrow{q} + \overrightarrow{r} + \overrightarrow{s} = 0$ $\overrightarrow{p} + \overrightarrow{q} = -\left(\overrightarrow{s} + \overrightarrow{r}\right)$	**B**	**1**

Question 6 (1 mark)

Outcomes assessed: ME12-1, ME12-4

Targeted Performance Bands: E3

Solution	**Answer**	**Mark**
	B	**1**

Question 7 (1 mark)

Outcomes Assessed: ME11-2

Targeted Performance Bands: E4

Solution	**Answer**	**Mark**
$\frac{2}{1-3^x} > -1$ Let $u = 1-3^x$ $\frac{2}{u} > -1$ multiply both sides by u^2 $2u > -u^2$ $2u + u^2 > 0$ $u(2+u) > 0$ $u < -2$ or $u > 0$ $1-3^x < -2$ or $1-3^x > 0$ $1-3^x < -2 \Rightarrow 3^x > 3$ $1-3^x > 0 \Rightarrow 3^x < 1$ $\therefore\ x < 0$ or $x > 1$	**D**	**1**

Question 8 (1 mark)

Outcomes Assessed: ME11-5

Targeted Performance Bands: E3

Solution	**Answer**	**Mark**
2 blue marbles chosen in ${}^5C_2$ ways. One non-blue marble chosen in ${}^9C_1$ ways. 3 Marbles chosen in ${}^{14}C_3$ ways. $\therefore \text{Prob}(\text{excatly 2 blue marbles}) = \frac{{}^5C_2 \times {}^9C_1}{{}^{14}C_3}$	**B**	**1**

Question 9 (1 mark)

Outcomes Assessed: ME12-1

Targeted Performance Bands: E3-E4

Solution	Answer	Mark
$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{4-4x^2}}\,dx = \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$ $= \frac{1}{2}\left[\sin^{-1} x\right]_0^{\frac{1}{2}}$ $= \frac{1}{2}\left[\frac{\pi}{6} - 0\right]$ $= \frac{\pi}{12}$	**B**	**1**

Question 10 (1 mark)

Outcomes assessed: ME11-4

Targeted Performance Bands: E4

Solution	Answer	Mark
$\frac{dP}{dt}$ has maximum value when $P = 50$ $\therefore$ population is increasing most rapidly when $P = 50$.	**B**	**1**

Question 11 (15 marks)

(a) (i) (1 mark)

Outcomes assessed: ME11-1

Targeted Performance Bands: E2

Criteria	Marks
• Correctly shows the result.	**1**

Sample answer:

$$P(1) = 1 - 5 + 8 - 4 = 0$$

(a) (ii) (2 marks)

Outcomes assessed: ME11-1

Targeted Performance Bands: E2

Criteria	Marks
• Correct factorisation.	**2**
• Recognises that $(x-1)$ is a factor of $P(x)$ and makes some attempt to factorise.	**1**

Sample answer:

$$(x-1)\overline{\big)x^3-5x^2+8x-4}\quad \text{quotient: } x^2-4x+4$$

$$P(x)=(x-1)(x-2)^2$$

(b) (2 marks)

Outcomes assessed: ME11-3

Targeted Performance Bands: E3

Criteria	Marks
• Correct Range	**2**
• Correct Domain.	**1**

Sample answer:

$$D_{f^{-1}}=R_f \Rightarrow : D_{f^{-1}} : 0<x\le 1$$

$$R_{f^{-1}}=D_f \Rightarrow R_{f^{-1}} : y\ge 0$$

(c) (2 marks)

Outcomes assessed: ME11-3

Targeted Performance Bands: E3

Criteria	Marks
- Correct answer.	**2**
- Finds the value of $\sin x$ or equivalent progress.	**1**

Sample answer:

$$\cos x = -\frac{1}{3}$$

$$\sin^2 x = 1 - \cos^2 x = \frac{8}{9}$$

$$\sin x = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3} \ (\text{given } \sin x > 0)$$

$$\sin 2x = 2\sin x \cos x = 2 \times \left(-\frac{1}{3}\right) \times \frac{2\sqrt{2}}{3} = -\frac{4\sqrt{2}}{9}$$

(d) (i) (2 marks)

Outcomes assessed: ME11-5, ME12-7

Targeted Performance Bands: E3

Criteria	Marks
- Provides correct explanation.	**2**
- Identifies that there are 5 pairs that give a sum of 11.	**1**

Sample answer:

Each number has a "partner" to give a total of 11
There are 5 pairs that give a total of 11:
$(1,10);(2,9);(3,8);(4,7);(5,6)$
That is: 5 pigeonholes
$\therefore$ by the pigeonhole principle, if 6 players are chosen
there will be at least one pair that gives a sum of 11.

(d) (ii) (1 mark)

Outcomes assessed: ME11-5, ME12-7

Targeted Performance Bands: E3

Criteria	Marks
• Correct answer	**2**
• Recognises that there are some numbers that don't have "pair" to sum to 12.	**1**

Sample answer:

The numbers "1" and "6"do not have a "partner" to give a total of 12

There are 4 pairs that give a total of 12:

$(2,10);(3,9);(4,8);(5,7)$

Will need to choose 7 players.

(e) (i) (3 marks)

Outcomes assessed: ME12-2

Targeted Performance Bands: E3-E4

Criteria	Marks
• Correct answer.	**3**
• Correctly determines initial vertical and horizontal velocities.	**2**
• Makes some progress (e.g. makes some attempt to find initial velocities.	**1**

Sample answer:

$$x = 30\sqrt{3}\,t \Rightarrow \frac{dx}{dt} = 30\sqrt{3}$$

$$y = 30t - 5t^2 \Rightarrow \frac{dy}{dt} = 30 - 10t$$

$$\text{when } t = 0,\ \frac{dy}{dt} = 30$$

if θ is the angle of projection, then

$$\tan\theta = \frac{30}{30\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\theta = 30^\circ$$

(e) (ii) (1 mark)

Outcomes assessed: HE3

Targeted Performance Bands: E3-E4

Criteria	Marks
- Correct answer	**1**

Sample answer:

If initial speed is V
then
$V^2 = 30^2 + \left(30\sqrt{3}\right)^2$
$V = 60\,ms^{-1}$

Question 12 (15 marks)

(a) (2 marks)

Outcomes Assessed: ME11-3

Targeted Performance Bands: E2-E3

Criteria	Marks
- Correct answer.	**2**
- Finds correct domain or range or equivalent.	**1**

Sample answer:

For $y = 2\cos^{-1}(2x-3)$
$D: -1 \le 2x - 3 \le 1$
$\quad 2 \le 2x \le 4$
$\quad 1 \le x \le 2$
$R: 0 \le y \le 2\pi$
$\therefore A\ is\ \left(1, 2\pi\right)$

(b) (i) (1 mark)

Outcomes assessed: ME12-2

Targeted Performance Bands: E2

Criteria	Marks
- Correct answer.	**1**

Sample answer:

$$Parallel \Rightarrow \lambda\begin{pmatrix} a \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$3\lambda = 4 \Rightarrow \lambda = \frac{4}{3}$$

$$\frac{4}{3}a = 2$$

$$a = \frac{3}{2}$$

(b) (ii) (3 marks)

Outcomes assessed: ME12-2

Targeted Performance Bands: E3

Criteria	Marks
- Correct answer	**3**
- Writes correct expression for $\left\|\vec{q} - \vec{r}\right\|$	**2**
- Recognises that the dot product is zero.	**1**

Sample answer:

$$\text{Perpendicular} \Rightarrow \lambda \begin{pmatrix} 2 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$2x + 4y = 0$$

$$x = -2y$$

$$\vec{q} - \vec{r} = \begin{pmatrix} 2-x \\ 4-y \end{pmatrix} = \begin{pmatrix} 2+2y \\ 4-y \end{pmatrix}$$

$$\left|\vec{q} - \vec{r}\right| = \sqrt{65} \Rightarrow \sqrt{(2+2y)^2 + (4-y)^2} = \sqrt{65}$$

squaring

$$4 + 8y + 4y^2 + 16 - 8y + y^2 = 65$$

$$5y^2 + 20 = 65$$

$$y^2 = 9$$

$$y = \pm 3$$

$$\therefore\ x = 6, y = -3 \text{ or } x = -6, y = 3$$

(c) (i) (1 mark)

Outcomes assessed: ME12-5

Targeted Performance Bands: E2

Criteria	Marks
• Correct answer.	**1**

Sample answer:

$$E(X) = np = 10\,000 \times 0.8 = 8000$$

(c) (ii) (1 mark)

Outcomes assessed: ME12-5

Targeted Performance Bands: E2

Criteria	Marks
• Correctly shows the result	**1**

Sample answer:

$$\begin{aligned} Var(X) &= np(1-p) \\ &= 10000 \times 0.8 \times 0.2 \\ &= 1600 \end{aligned}$$

$$\sigma = \sqrt{1600} = 40$$

(c) (iii) (1 mark)

Outcomes assessed: ME12-5

Targeted Performance Bands: E3

Criteria	Marks
- Correct answer.	**1**

Sample answer:

$$\begin{aligned} P(X<8040) &= P(Z<1) \\ &= 0.5+0.34 \\ &= 0.84 \end{aligned}$$

(d) 3 marks

Outcomes assessed: ME12-1, ME12-4

Targeted Performance Bands: E3

Criteria	Marks
- Correct answer.	**3**
- Correct integration	**2**
- Separates the differential equation and makes some attempt to integrate.	**1**

Sample answer:

$$\frac{dy}{dx} = 2xy$$

$$\frac{1}{y}dy = 2xdx$$

$$\int \frac{1}{y}dy = \int 2xdx$$

$$\log_e y = x^2 + c$$

Passes through $(0,4)$ $\Rightarrow \text{c}=\log_e 4$

$$\log_e y = x^2 + \log_e 4$$

$$\log_e y - \log_e 4 = x^2$$

$$\log_e \frac{y}{4} = x^2$$

$$y = 4e^{x^2}$$

(b) (3 marks)

Outcomes assessed: ME12-1; ME12-7

Targeted Performance Bands: E3 – E4

Criteria	Marks
• Correctly proves the result.	**3**
• Uses the correct assumption step in attempting to prove the statement is true for $n = k + 1$	**2**
• Correctly proves that the statement is true when $n = 1$.	**1**

Sample answer:

Step 1 Prove true for $n = 1$

$$LHS = \frac{1}{2}$$

$$RHS = 2 - \frac{1+2}{2} = \frac{1}{2}$$

$\therefore$ true for $n = 1$

Step 2 Let $n = k$ be a value for which the statement is true

ie

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + ... + \frac{k}{2^k} = 2 - \frac{k+2}{2^k}$$

Step 3 Prove true for $n = k + 1$

i.e prove

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + ... + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} = 2 - \frac{(k+1)+2}{2^{k+1}} = 2 - \frac{k+3}{2^{k+1}}$$

$$\begin{aligned} LHS &= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}} \\ &= 2 - \left(\frac{2(k+2)}{2^{k+1}} - \frac{k+1}{2^{k+1}} \right) \\ &= 2 - \left(\frac{2k+4-k-1}{2^{k+1}} \right) \\ &= 2 - \frac{k+3}{2^{k+1}} \text{ as required.} \end{aligned}$$

$\therefore$ true by mathematical induction

Question 13 (15 marks)

(a) (2 marks)

Outcomes assessed: ME12-1

Targeted Performance Bands: E3

Criteria	**Marks**
- Correct answer	**2**
- Chooses appropriate substitution.	**1**

Sample answer:

$$Let\ u = \sin x$$

$$\int_0^{\frac{\pi}{2}} \sin^2 x \cos x \, dx = \int_0^1 u^2 \, du$$

$$= \left[\frac{u^3}{3} \right]_0^1 = \frac{1}{3}$$

(b) (3 marks)

Outcomes Assessed: ME12-4

Targeted Performance Bands: E4

Criteria	**Marks**
- correct answer	**3**
- correct final integral	**2**
- any correct attempt	**1**

Sample answer:

$x = 2\sin\theta$

$\dfrac{dx}{d\theta} = 2\cos\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \dfrac{\pi}{6}$

$$\begin{aligned}\int_0^1 \sqrt{4-x^2}\,dx &= \int_0^{\frac{\pi}{6}} \sqrt{4-4\sin^2\theta} \times 2\cos\theta\,d\theta \\ &= \int_0^{\frac{\pi}{6}} 2\cos\theta \times 2\cos\theta\,d\theta \\ &= 4\int_0^{\frac{\pi}{6}} \cos^2\theta\,d\theta \\ &= 4\int_0^{\frac{\pi}{6}} \frac{1}{2}\left(1+\cos 2\theta\right)d\theta \\ &= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}} \\ &= 2\left[\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\right]_0^{\frac{\pi}{6}} \\ &= \frac{\pi}{3} + \frac{\sqrt{3}}{2}\end{aligned}$$

(c) (i) (3 marks)

Outcomes Assessed: ME12-3

Targeted Performance Bands: E4

Criteria	Marks
• Correctly shows the result.	**3**
• Makes some correct use of $\cos 2x$ results or equivalent progress.	**2**
• Makes some use of "products to sums" results or equivalent progress.	**1**

Sample answer:

$$\begin{aligned}\sin(x-y)\sin(x+y) &= \frac{1}{2}\Big[\cos\big[(x-y)-(x+y)\big]-\cos\big[(x-y)+(x+y)\big]\Big] \\ &= \frac{1}{2}\big[\cos(-2y)-\cos(2x)\big] \\ &= \frac{1}{2}\big[\cos(2y)-\cos(2x)\big] \\ &= \frac{1}{2}\Big[\left(1-2\sin^2 y\right)-\left(1-2\sin^2 x\right)\Big] \\ &= \sin^2 x-\sin^2 y\end{aligned}$$

(c) (ii) (3 marks)

Outcomes Assessed: ME12-3

Targeted Performance Bands: E4

Criteria	Marks
• Correctly solves the equation.	**3**
• Finds one correct non-zero solution.	**2**
• Correctly uses the result from (i) to transform equation.	**1**

Sample answer:

From (i) $\sin^2 3x - \sin^2 x = \sin(3x-x)\sin(3x+x) = \sin 2x \sin 4x$

$\sin 2x \sin 4x = \sin 4x \ \ for \ \ 0 \le x \le \pi$

$\sin 2x \sin 4x - \sin 4x = 0$

$\sin 4x(\sin 2x - 1) = 0$

$\sin 4x = 0 \Rightarrow 4x = 0, \pi, 2\pi, 3\pi, 4\pi$

$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

$\sin 2x = 1 \Rightarrow 2x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{4}$

$\therefore x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

(d) (4 marks)

Outcomes Assessed: ME12-4

Targeted Performance Bands: E4

Criteria	**Marks**
• Provides correct answer.	**4**
• Makes further progress finding integral.	**3**
• Writes correct integral expression for the volume.	**2**
• Finds correct points of intersection	**1**

Sample answer:

For point of intersection

$x+1=(x-1)^2$

$x+1=x^2-2x+1$

$x^2-3x=0$

$x=0, x=3$

$$V=\pi\int_0^3 (x+1)^2-\left[(x-1)^2\right]^2 dx$$

$$=\pi\int_0^3 (x+1)^2-(x-1)^4 dx$$

$$=\pi\left[\frac{(x+1)^3}{3}-\frac{(x-1)^5}{5}\right]_0^3$$

$$=\pi\left[\left(\frac{64}{3}-\frac{32}{5}\right)-\left(\frac{1}{3}-\frac{1}{5}\right)\right]$$

$$=\frac{74\pi}{5}\ units^3$$

Question 14 (15 marks)

Sample answer:

(a) i) $(1+x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$

As m > n this means $(1+x)^m$ has more terms than $(1+x)^n$.

$(1+x)^m = \binom{m}{0}x^0 + \binom{m}{1}x^1 + \binom{m}{2}x^2 + \cdots + \binom{m}{n}x^n \ldots\ldots + \binom{m}{m-1}x^{m-1} + \binom{m}{m}x^m$

In the expansion of $(1+x)^n(1+x)^m$, each term in the expansion of $(1+x)^n$ will be multiplied by each term in the expansion of $(1+x)^m$.

To show that the required statement is true, we must equate the coefficients of x^n in the expansions of $(1+x)^n(1+x)^m$ and $(1+x)^{n+m}$

The coefficient of x^n in the expansion of $(1+x)^n(1+x)^m$ is found by multiplying each term in the expansion of $(1+x)^n$ by the corresponding term in the expansion of $(1+x)^m$, such that the sum of the powers of x is n.

Hence, the corresponding coefficients of x^n will be

$\binom{n}{0}\binom{m}{n} + \binom{n}{1}\binom{m}{n-1} + \binom{n}{2}\binom{m}{n-2} + \binom{n}{3}\binom{m}{n-3} + \cdots + \binom{n}{n-1}\binom{m}{1} + \binom{n}{n}\binom{m}{0}$

In the expansion of $(1+x)^{n+m}$, the coefficient of x^n will be $\binom{m+n}{n}$. Hence,

$\binom{n}{0}\binom{m}{n} + \binom{n}{1}\binom{m}{n-1} + .. + \binom{n}{n}\binom{m}{0} = \binom{m+n}{n}$

ii) The number of different combinations possible for n canaries from n white and m yellow canaries is

$\binom{n}{0}\binom{m}{n} + \binom{n}{1}\binom{m}{n-1} + .. + \binom{n}{n}\binom{m}{0}$

From the previous part of this question this would be equivalent to :- $\binom{m+n}{n}$

As Simon must have at least one canary of each colour, no white and n yellow $\binom{n}{0}\binom{m}{n}$, n white and no yellow $\binom{n}{n}\binom{m}{0}$ are not acceptable. Hence, the number of ways Simon could select n canaries, ensuring there was at least one of each colour is

$\binom{m+n}{n} - \binom{n}{0}\binom{m}{n} - \binom{n}{n}\binom{m}{0}$ As $\binom{n}{n} = 1$, $\binom{m}{0} = 1$ *and* $\binom{n}{0} = 1$ then the number of ways is

$\binom{m+n}{n} - 1 \times \binom{m}{n} - 1 \times 1 = \binom{m+n}{n} - \binom{m}{n} - 1$

b) i) $\overrightarrow{AM} = \underset{\sim}{a} + \overrightarrow{BM}$ (1)

also $\overrightarrow{AM} = \underset{\sim}{b} + \overrightarrow{CM}$ and as

$\overrightarrow{CM} = -\overrightarrow{BM}$ then

$\overrightarrow{AM} = \underset{\sim}{b} - \overrightarrow{BM}$ (2)

By adding (1) and (2) , we get

$2\overrightarrow{AM} = \underset{\sim}{a} + \underset{\sim}{b}$ that is

$$\overrightarrow{AM} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$$

$$\overrightarrow{CM} = -\underset{\sim}{b} + \overrightarrow{AM}$$

$$\overrightarrow{CM} = -\underset{\sim}{b} + \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$$

$$\overrightarrow{CM} = -\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$$

$$\overrightarrow{CM} = \frac{1}{2}\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b}$$

$$\overrightarrow{CM} = \frac{1}{2}(\underset{\sim}{a} - \underset{\sim}{b})$$

ii) LHS $= \left|\overrightarrow{AM}\right|^2 + \left|\overrightarrow{CM}\right|^2$

$$= \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}) \cdot \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}) + \frac{1}{2}(\underset{\sim}{a} - \underset{\sim}{b}) \cdot \frac{1}{2}(\underset{\sim}{a} - \underset{\sim}{b})$$

$$= \frac{1}{4}(\underset{\sim}{a} + \underset{\sim}{b})^2 + \frac{1}{4}(\underset{\sim}{a} - \underset{\sim}{b})^2$$

$$= \frac{1}{4}(|\underset{\sim}{a}|^2 + 2\underset{\sim}{a}\underset{\sim}{b} + |\underset{\sim}{b}|^2 + |\underset{\sim}{a}|^2 - 2\underset{\sim}{a}\underset{\sim}{b} + |\underset{\sim}{b}|^2)$$

$$= \frac{1}{4}(2|\underset{\sim}{a}|^2 + 2|\underset{\sim}{b}|^2)$$

$$= \frac{1}{2}(|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2)$$

$$= \frac{1}{2}\left(\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AC}\right|^2\right)$$

$$= RHS$$

c) i) $cos4\theta = 2cos^2 2\theta - 1$

$cos4\theta = 2(2cos^2\theta - 1)^2 - 1$

$cos4\theta = 2(4cos^4\theta - 4cos^2\theta + 1) - 1$

$cos4\theta = 8cos^4\theta - 8cos^2\theta + 2 - 1$

$cos4\theta = 8cos^4\theta - 8cos^2\theta + 1$

$0 = 8cos^4\theta - 8cos^2\theta + 1 - cos4\theta$

ii) $cos4\theta = 8cos^4\theta - 8cos^2\theta + 1$

$2cos4\theta = 16cos^4\theta - 16cos^2\theta + 2$

By substituting

$16cos^4\theta - 16cos^2\theta + 2 - \sqrt{2} = 0$

from above, we get

$2cos4\theta - \sqrt{2} = 0$

$2cos4\theta = \sqrt{2}$

$cos4\theta = \frac{\sqrt{2}}{2}$

iii) From part ii) the solution to the equation

$16cos^4\theta - 16cos^2\theta + 2 - \sqrt{2} = 0$ can be found

from $\cos 4\theta = \frac{\sqrt{2}}{2}$

$\cos 4\theta = \cos \frac{\pi}{4}$

$$4\theta = \frac{\pi}{4} + 2n\pi \;\; or \;\; 4\theta = -\frac{\pi}{4} + 2n\pi$$

$$\theta = \frac{\pi}{16} + \frac{n\,\pi}{2} \quad or \quad \theta = -\frac{\pi}{16} + \frac{n\,\pi}{2}$$

$when\ n = 0\ , \theta = \frac{\pi}{16}$ or $when\ n = 0\ , \theta = -\frac{\pi}{16}$

$when\ n = 1\ , \theta = \frac{9\,\pi}{16}$ or $when\ n = 1, \theta = \frac{7\,\pi}{16}$

$when\ n = 2\ , \theta = \frac{17\,\pi}{16}$ or $when\ n = 2, \theta = \frac{15\pi}{16}$

$when\ n = 3\ , \theta = \frac{25\,\pi}{16}$ or $when\ n = 3, \theta = \frac{23\pi}{16}$

There is an infinite number of possible values for the angle θ. The four lines above show a few of these possible values.

As the equation $16\,x^4 - 16\,x^2 + 2 - \sqrt{2} = 0$ has a degree 4 it should have at maximum four real roots which mean four possible values for $\cos\theta$ regardless of the possible values of θ.

Hence, we need to find four different values for $\cos\theta$ using the possible values for angle θ found above as they are the four roots of this equation.

The four different values of $\cos\theta$ or the four roots of the quartic equation are

$\cos\frac{\pi}{16}, \cos\frac{15\pi}{16}, \cos\frac{7\pi}{16}$ and $\cos\frac{23\pi}{16}$

The product of these roots is $\frac{2-\sqrt{2}}{16}$.

This means

$\cos\frac{\pi}{16}\cos\frac{15\pi}{16}\,\cos\frac{7\pi}{16}\cos\frac{23\pi}{16} = \frac{2-\sqrt{2}}{16}$

Now, as $\cos\frac{15\pi}{16} = -\cos\frac{\pi}{16}$ and $\cos\frac{23\pi}{16} = -\cos\,\frac{7\pi}{16}$ this means

$\cos\frac{\pi}{16} \times -\cos\frac{\pi}{16}\,\cos\frac{7\pi}{16} \times -\cos\frac{7\pi}{16} = \frac{2-\sqrt{2}}{16}$

Hence, $cos^2\,\frac{\pi}{16}\,cos^2\,\frac{7\pi}{16} = \frac{2-\sqrt{2}}{16}$

Alternative method

From part ii) the solution to the equation

$16cos^4\theta - 16cos^2\theta + 2 - \sqrt{2} = 0$ can be found

from $\cos 4\theta = \frac{\sqrt{2}}{2}$

So $4\theta = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$ ....

$\therefore\ \theta = \frac{\pi}{16}, \frac{7\pi}{16}, \frac{9\pi}{16}, \frac{15\pi}{16}$ .... are solutions to the equation.

But $16cos^4\theta - 16cos^2\theta + 2 - \sqrt{2} = 0$ can be reduced to the quadratic

$16m^2 - 16m + 2 - \sqrt{2} = 0$ where $m = cos^2\theta$

As $\theta = \frac{\pi}{16}, \frac{7\pi}{16}, \frac{9\pi}{16}, \frac{15\pi}{16}$ .... are solutions to the initial trig equation, $cos^2\frac{\pi}{16}$ $and\ cos^2\frac{7\pi}{16}$ will be roots of this new equation.

The product of the roots of a quadratic equation is equal to the constant term, divided by the coefficient of m^2,

Hence, $cos^2\frac{\pi}{16}.cos^2\frac{7\pi}{16}=\frac{2-\sqrt{2}}{16}$